ODD GRACEFULL LABELING FOR THE SUBDIVISON OF DOUBLE TRIANGLES GRAPHS

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ABSTRACT

The aim of this paper is to present some odd graceful graphs. In particular we show that an odd graceful labeling of the all subdivision of double triangular snakes ($2\Delta_k$ -snake). We also prove that the all subdivision of $2m\Delta_1$-snake are odd graceful. Finally, we generalize the above two results (the all subdivision of $2m\Delta_k$ -snake are odd graceful).

KEYWORDS

Graph Labeling; Odd Graceful; Subdivision; Triangular Snakes

1.INTRODUCTION

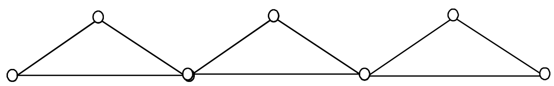
The graphs considered here will be finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G respectively. p and q denote the number of vertices and edges of G respectively.

A graph G of size q is odd-graceful, if there is an injection ϕ from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label or weight $|\phi(x) - \phi(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. This definition was introduced in 1991 by Gnanajothi [1] who proved that the class of odd graceful graphs lies between the class of graphs with -labelings and the class of bipartite graphs.

Rosa [2] defined a triangular snake(or Δ-*snake*) as a connected graph in which all blocks are triangles and the block-cut-point graph is a path. Let Δ_k -*snake* be a Δ-*snake* with k

blocks while $n\Delta_k$ *-snake* be a Δ *-snake* with k blocks and every block has n number of triangles with one common edge.

Illustration1:



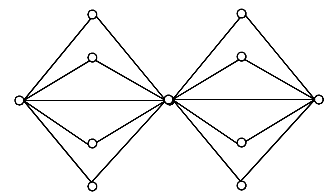


Figure 1: Δ_3-snake $\quad$ $4\Delta_2$ -snake

Rosa[2] conjectured that Δ_k *-snake* (a snake with k blocks) is graceful for $n \equiv 0$ or 1 (mod 4) and is nearly graceful otherwise. In 1989 Moulton [3] has proved Rosa's conjecture but using instead of nearly graceful labeling an stronger labeling named almost graceful.

A double triangular snake is a graph that formed by two triangular snakes have a common path. The harmonious labeling of double triangle snake introduced by Xi Yue et al [4]. It is known that the graphs which contain odd cycles are not odd graceful so Badr [5] used the subdivision notation for odd cycle in order to prove that the subdivision of linear triangular snakes are odd graceful. Badr et al [6] proved that the subdivision of ladders $S(L_n)$ is odd graceful.

In this paper we prove that an odd graceful labeling of the all subdivision of double triangular snakes ($2\Delta_k$ *-snake*). We also prove that the all subdivision of $2m\Delta_1$*-snake* are odd graceful. Finally, we generalize the above two results (the all subdivision of $2m\Delta_k$ *-snake* are odd graceful).

2. MAIN RESULTS

***Theorem* 2.1** All the subdivision of double triangular snakes ($2\Delta_k$ *-snake*) are odd graceful.

Proof: Let $G = 2\Delta_k$ *-snake* has q edges and p vertices. The graph G consists of the vertices ($u_1, u_2, \ldots, u_{k+1}$), ($v_1, v_2, \ldots, v_k$), ($w_1, w_2, \ldots, w_k$) therefore we get the subdivision of double triangular snakes $S(G)$ by subdividing every edge of double triangular snakes $2\Delta_k$ *-snake* exactly once. Let y_i be the newly added vertex between u_i and u_{i+1} while w_{i1} and w_{i2} are newly added vertices between $w_i\, u_i$ and $w_i\, u_{i+1}$ respectively, where $1 \le i \le k$. Finally,v_{i1} and v_{i2} are newly added vertices between $v_i\, u_i$ and $v_i\, u_{i+1}$respectivley,such that $1 \le i \le k$.
The graph $S(G)$ consists of the vertices ($u_1, u_2, \ldots, u_{k+1}$), ($v_1, v_2, \ldots, v_k$), ($w_1, w_2, \ldots, w_k$), ($w_{11}, w_{12}, w_{21}, w_{22}, \ldots, w_{k1}, \ldots, w_{k2}$),($v_{11}, v_{12}, v_{21}, v_{22}, \ldots, v_{k1}, v_{k2}$) and ($y_1, y_2, \ldots, y_k$) as shown in Figure 2. Clearly $S(G)$ has $q = 10k$ edges and $p = 8k + 1$ vertices.

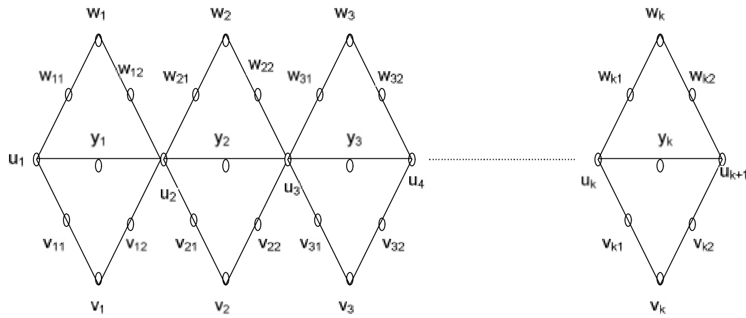


Figure 2: the subdivision of double triangular snakes ($2\Delta_k$ *-snake*)

we prove that the subdivision of double triangular snakes $S(G)$ is odd graceful.
Let us consider the following numbering ϕ of the vertices of the graph G :

$\phi\ (u_i) = 6(\ i - 1\ )$ $\quad 1 \leq i \leq k+1$
$\phi\ (y_i) = 2q - 14i + 11$ $\quad 1 \leq i \leq k$
$\phi\ (v_i) = 6i - 4$ $\quad 1 \leq i \leq k$
$\phi\ (v_{ij}) = 2q - 14i - 8j + 17$ $\quad 1 \leq i \leq k,\ j = 1, 2$
$\phi\ (w_i) = 6i + 4$ $\quad 1 \leq i \leq k$
$\phi\ (w_{ij}) = 2q - 14i - 6j + 19$ $\quad 1 \leq i \leq k,\ j = 1, 2$

(a) $$\underset{v \in V}{Max}\ \phi(v) = \max\left\{ \max_{1 \leq i \leq k+1} 6(\ i - 1\ ), \max_{1 \leq i \leq k} 2q - 14i + 11\ - 1, \max_{1 \leq i \leq k} 6i - 4, \max_{\substack{1 \leq j \leq 2 \\ 1 \leq i \leq k}} 2q - 14i - 8j + 17, \right.$$
$$\left. \max_{1 \leq i \leq k} 6i + 4, \max_{\substack{1 \leq j \leq 2 \\ 1 \leq i \leq k}} 2q - 14i - 6j + 19 \right\}$$

= $2q - 1$, the maximum value of all odds.

Thus $\phi\ (v) \in \{0, 1, 2, \ldots, 2q - 1\}$.

(b) Clearly ϕ is a one – to – one mapping from the vertex set of G to $\{0, 1, 2, \ldots, 2q-1\}$.

(c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1, 2q-1]$.

The range of $\phi\ (u_i) - \phi\ (w_{i1}) = \{2q - 20i + 19;\ i = 1,2,\ldots,k\} = \{2q-1, 2q-21, \ldots, 19\}$
The range of $\phi\ (u_i) - \phi\ (y_i) = \{2q - 20i + 17;\ i = 1,2,\ldots,k\} = \{2q-3, 2q-23, \ldots, 17\}$
The range of $\phi\ (u_i) - \phi\ (v_{i1}) = \{2q-20i+15;\ i = 1,2,3,\ldots, k\} = \{2q-5, 2q-25, \ldots, 15\}$
The range of $\phi\ (v_{i1}) - \phi\ (v_1) = \{2q-20i+13;\ i = 1,2,3,\ldots, k\} = \{2q-7, \ldots, 13\}$
The range of $\phi\ (y_i) - \phi\ (u_{i+1}) = \{2q-20i+11;\ i = 1,2,3,\ldots, k\} = \{2q-9, \ldots, 11\}$

The range of ϕ (w_{i1})- ϕ (w_i) $=\{2q - 20i + 9 ; i =1,2,\ldots, k\} = \{2q - 11,\ldots,9\}$
The range of ϕ (w_{i2})- ϕ (u_{i+1}) $=\{2q - 20i + 7 ; i =1,2,\ldots, k\} = \{2q - 13,\ldots,7\}$
The range of ϕ (v_i)- ϕ (v_{i2}) $= \{2q - 20i + 5 ; i =1,2,3,\ldots, k\} = \{2q - 15,\ldots,5\}$
The range of ϕ (w_{i2})- ϕ (w_i) $=\{2q - 20i + 3 ; i =1,2,3,\ldots, k\} = \{2q - 17,\ldots,3\}$
The range of ϕ (v_{i2})- ϕ (u_{i+1}) $=\{2q - 20i + 1 ; i =1,2,3,\ldots, k\} = \{2q -19,\ldots,1\}$
Hence { ϕ (u) - ϕ (v) : $uv \in E$ } = {1, 3, 5,…, $2q$-1}so that the subdivision of double triangular snakes ($2\Delta_k$ *-snake*) are odd graceful.

Illustration2:

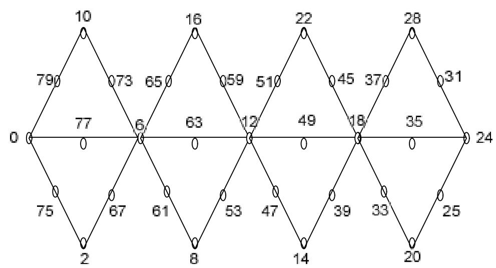


Figure 3: odd-graceful labeling of the graph $S(2\Delta_4$ *-snake*).

***Theorem* 2.2** All the subdivision of $2m\Delta_1$*-snake* are odd-graceful, where $m \geq 1$.

Proof :

Let $G = 2m\Delta_1$*-snake* has q edges and p vertices. The graph G consists of the vertices (u_1, u_2), ($v_1^1, v_1^2, \ldots, v_1^m$), ($w_1^1, w_1^2, \ldots, w_1^m$) therefore we get the subdivision of the graph G, $S(G)$, by subdividing every edge of the graph $G = 2m\Delta_k$ *-snake* exactly once. Let y_1 be the newly added vertex between u_1 and u_2 while w_{11}^i and w_{12}^i are newly added vertices between $w_1^i\, u_1$ and $w_1^i\, u_2$ respectively. Finally , v_{11}^i and v_{12}^i are newly added vertices between $v_1^i\, u_1$ and $v_1^i\, u_2$ respectively where i = 1, 2, . . ., m.
The graph $S(G)$ consists of the vertices (u_1, u_2), ($v_1^1, v_1^2, \ldots, v_1^m$), ($w_1^1, w_1^2, \ldots, w_1^m$), y_1 , (w_{11}^i, $w_{12}^i, \ldots, w_{12}^m$) and ($v_{11}^i, v_{12}^i, \ldots, v_{12}^m$) as shown in Figure 4. Clearly $S(G)$ has $q = 8m + 2$ edges and $p = 6m + 3$ vertices.

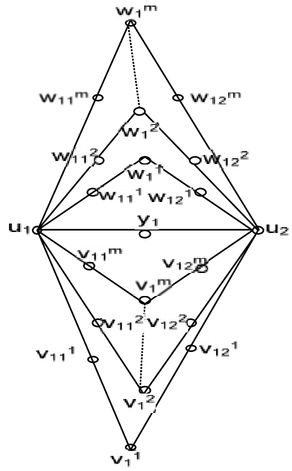


Figure 4: the subdivision of $2m\Delta_1$-*snake*

Let us consider the following numbering ϕ of the vertices of the graph G :

$\phi(u_i) = (4m + 2)(i - 1)$ $\quad i = 1,2$

$\phi(y_1) = 14m+3$

$\phi(v_i^l) = 4l - 2$ $\quad 1 \le l \le m$

$\phi(w_1^l) = 4(l + m) + 2$ $\quad 1 \le l \le m$

$\phi(v_{1j}^l) = 20m + 2l + 3 - (8m + 2)j$ $\quad 2 \le l \le m,\ j = 1,2$

$\phi(v_{1j}^1) = (18m+5) - (6m + 2)j$ $\quad j = 1,2$

$\phi(w_{1j}^l) = (18m+2l+5) - (4m+2)j$ $\quad j = 1,2,\ 2 \le l \le m$

The edge labels well be as follows:

The vertices u_1 and w_{11}^l, $1 \le l \le m$, induce the edge labels = $\{14m + 2l + 3 ,\ 1 \le l \le m\}$ $= \{14m + 5 , 14m + 7 ,\ldots,16m + 3\}$

The vertices u_1 and y_1 induce the edge labels $\{14m + 3\}$

The vertices u_1 and v_{11}^l; $1 \le l \le m$, induce the edge labels $\{12m + 3,\ 12m + 2l+ 1 ;\ 2 \le l \le m\}$ $=\{12m + 3,\ 12m + 5,\ldots,14m + 1\}$

The vertices v_{11}^l and v_1^l ,y_1 and u_2 , w_{11}^l and w_1^l induce the edge labels ; $1 \le l \le m$:
$\{12m + 3 - 2l: 1 \le l \le m\},\{10m + 1\},\{10m - 2l + 1;\ 1 \le l \le m\}=\{12m + 1,\ 12m-1,\ldots,10m +1,\ 10m - 1,\ldots,8m+1\}$.

The vertices w_{12}^l and u_2 induce the edge label$\{6m + 2l - 1\};\ 1 \le l \le m\}=\{6m + 1,\ 6m + 3,\ldots,\ 8m - 1\}$

The vertices v_1^1 and v_{12}^1 induce the edge label $\{6m - 1\}$.

The vertices w_{12}^l and w_i^l , induce the edge labels $\{6m - 2l - 1;\quad 1 \le l \le m\}=\{6m - 3,\ 6m-5,\ldots,4m-1\}$.

The vertices v_{12}^l and v_1^l ; $2 \le l \le m$ induce the edge labels = $\{4m - 2l + 1;\ 2 \le l \le m\}$ $= \{4m - 3,\ 4m-5,\ldots,2m+1\}$.

The vertices v_{12}^{1} and u_2 induce the edge label $\{2m - 1\}$.

Finally the vertices v_{12}^{l} and u_2 induce the edge labels$\{2l\text{-}3\ ;\ 2 \le l \le m\} = \{1,3,5,\ldots..,2m\text{-}3\}$.

Hence the graph $S(2m\Delta_1\text{-}snake)$ is odd-graceful for each $m \ge 1$.

***Theorem* 2.3** All subdivision of $2m\Delta_k$ *-snake* are odd-graceful

Proof.

Let $G=2m\Delta_k$ *-snake* has q edges and p vertices. The graph G consists of the vertices ($u_1, u_2,\ldots,u_{k+1}$), $(v_1^1, v_1^2, \ldots, v_1^m)$, $(v_2^1, v_2^2, \ldots, v_2^m)$, . . ., $(v_k^1, v_k^2, \ldots, v_k^m)$, $(w_1^1, w_1^2 \ldots, w_1^m)$, $(w_2^1, w_2^2 \ldots, w_2^m)$, . . ., $(w_k^1, w_k^2 \ldots, w_k^m)$ therefore we get the subdivision of double triangular snakes $S(G)$ by subdividing every edge of $2m\Delta_k$ *-snake* exactly once. Let y_1 be the newly added vertex between u_1 and u_2 while w_{i1}^{j} and w_{i2}^{j} are newly added vertices between $w_i^j\, u_i$ and $w_i^j\, u_{i+1}$ respectively. Finally , v_{i1}^{j} and v_{i2}^{j} are newly added vertices between $v_i^j\, u_i$ and $v_i^j\, u_{i+1}$ - respectively where $i = 1, 2, \ldots, k$ *and* $j = 1, 2, 3, \ldots, m$ (Figure 5). Clearly $S(G)$ has $q = k\,(8m + 2)$ edges.

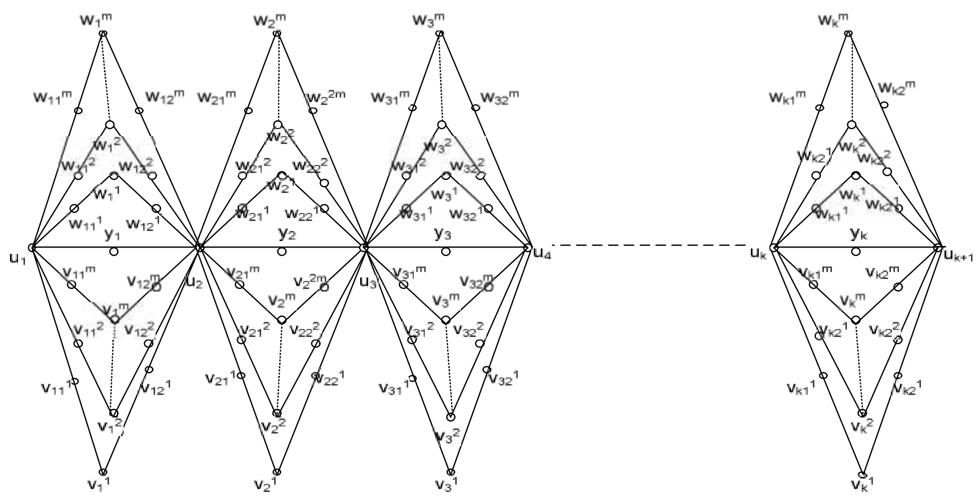


Figure 5: the subdivision of $2m\Delta_k$ *-snake*

Let us consider the following numbering ϕ of the vertices of the graph G :

$\phi\ (u_i) = (4m + 2)(\,i - 1)$ $\qquad 1 \le i \le k$

$\phi\ (w_i^l) = (4m+2)i + 4l$ $\qquad 1 \le i \le k\ ,\ 1 \le l \le m$

$\phi\ (v_i^l) = (4m+2)i + 4(l\text{-}m\text{-}1)$ $\qquad 1 \le i \le k\ ,\ 1 \le l \le m$

$\phi\ (\,y_i\,) = 2q\text{-}(12m+2)i + 10m+1\ ;\ 1 \le i \le k\ ,\ 1 \le j \le 2$

$\phi\ (w_{ij}^{\ l}) = 2q\text{-}(12m+2)i - (4m+2)j + 14m + 2l + 3;\ \ 1 \le i \le k\ ,\ 1 \le j \le 2,\ 1 \le l \le m$

$\phi\ (v_{ij}^{\ l}) = 2q\text{-}(12m+2)i - (8m+2)j + 16m + 2l + 1;\ \ 1 \le j \le 2,\ 1 \le l \le m$

$\phi\ (v_{ij}^{\ 1}) = 2q\text{-}(12m+2)i - (6m+2) + 14m + 3\ ;\ 1 \le i \le k\ ,\ 1 \le j \le 2$

In a view of the above defined labeling pattern ϕ is odd-graceful for the graph $S(G)$. hence $S(G)$ is odd-graceful for all $m \ge 1$, $k \ge 1$.

Illustration 2,3 :

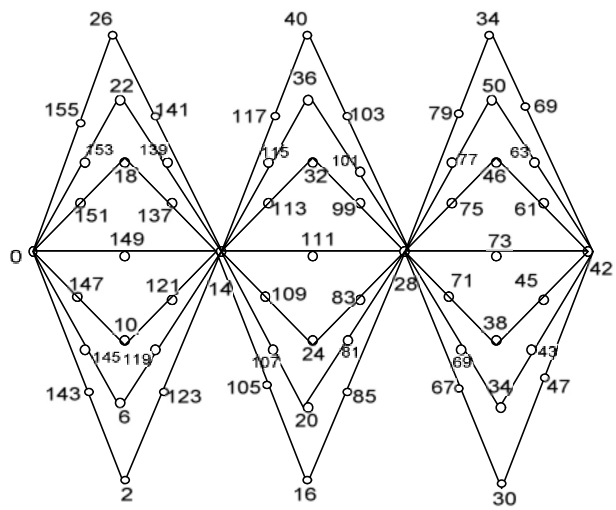


Figure 6: odd-graceful labeling of the graph $6\Delta_3$ -*snake*

3. **CONCLUSION**

Graceful and odd graceful of a graph are two entirely different concepts. A graph may posses one or both of these or neither. In the present work we show that an odd graceful labeling of the all subdivision of double triangular snakes ($2\Delta_k$ -*snake*). We also proved that the all subdivision of $2m\Delta_1$-*snake* are odd graceful. Finally, we generalized the above two results (the all subdivision of $2m\Delta_k$ -*snake* are odd graceful).

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